1- Approximate the solution to O.D.E $y' = y - t^2 + 1$, $0 \le t \le 0.6$

$y(0) = 0.5$ at $t = 0.6$ using

1) Euler method

2) 2^{nd} & 3^{rd} Taylor's method

3) 4th order RK (taking only one step) $y|_{t=0.3}$

4) obtain the global error of method ① with $h = 0.2$ in the interval $t \in [0, 2]$, knowing that the exact sol. is $y(t) = (t+1)^2 - 0.5e^t$ (or solve L.D.E).

Sol.

(1) Euler

$$y_{i+1} = y_i + h\,f(t_i, y_i)$$

$t_0 = 0$, $y_0 = 0.5$; $t_1 = 0.3$, $y_1 =$; $t_2 = 0.6$, $y_2 =$

$$\therefore y_{i+1} = y_i + h(y_i - t_i^2 + 1)$$

i = 0

$$y_1 = y_0 + h(y_0 - t_0^2 + 1) = 0.5 + 0.3(0.5 - 0 + 1)$$

$$\therefore y_1 = 0.95$$

i = 1

$$y_2 = y_1 + h(y_1 - t_1^2 + 1) = 0.95 + 0.3(0.95 - (0.3)^2 + 1)$$

$$y_2 = 1.508$$

(2) 2^{nd} order Taylor

$$y_{i+1} = y_i + h\,T^{(2)}(t_i, y_i)$$

$$T^{2}(t_i, y_i) = f(t_i, y_i) + \frac{h}{2} f'(t_i, y_i)$$

$$f(t_i, y_i) = y_i - t_i^2 + 1 \Rightarrow f'(t_i, y_i) = y_i' - 2t_i = y_i - 2t_i - t_i^2 + 1$$

$$\therefore T^{(2)}(t_i, y_i) = y_i - t_i^2 + 1 + \frac{h}{2}[y_i - 2t_i - t_i^2 + 1]$$

$$\therefore T_i^{(2)} = (1 + \frac{h}{2})[y_i - t_i^2 + 1] - h t_i$$

$i=0$

$$T_0^{(2)}(0,0.5) = \boxed{1.725}$$

$$\therefore y_1 = 0.5 + 0.3(1.725) \Rightarrow y_1 = 1.0175$$

$i=1$

$$\boxed{T_1^{(2)} = 2.126625}$$

$$y_2 = 1.0175 + 0.3(2.126625) \Rightarrow y_2 = 1.6554875$$

(2) 3rd order Taylor

$$y_{i+1} = y_i + hT^{(3)}(t_i, y_i)$$

$$T_i^{(3)} = f(t_i, y_i) + \frac{h}{2} f'(t_i, y_i) + \frac{h^2}{3!} f''(t_i, y_i)$$

$$f'(t_i, y_i) = y - 2t - t^2 + 1 \Rightarrow f_i'' = y' - 2t - 2 = y_i - t_i^2 - 2t_i - 1$$

$$\therefore T_i^{(3)} = T_i^{(2)} + \frac{h^2}{3!}[y_i - t_i^2 - 2t_i - 1]$$

$i=0$

$$T_0^{(3)} = T_0^{(2)} + \frac{h^2}{6}(y_0 - t_0^2 - 2t_0 - 1) = \boxed{1.7175}$$

$$\therefore y_1 = 0.5 + 0.3(1.7175) \Rightarrow y_1 = 1.01525$$

$i=1$

$$T_1^{(3)} = T_1^{(2)} + \frac{h^2}{6}(y_1 - t_1^2 - 2t_1 - 1)$$

$$\boxed{T_1^{(3)} = 2.11650375}$$

$$\therefore y_2 = y_1 + 0.3\,T_1^{(3)} \Rightarrow y_2 = 1.650201125$$

(3) 4^{th} order RK. (one-Step).

$$y_{i+1} = y_i + \frac{h}{6}(k_1+2k_2+2k_3+k_4)$$

$i=0$

$$k_1 = f(t_0, y_0) = f(0, 0.5) = \boxed{1.5}$$

$$k_2 = f(t_0+\tfrac{h}{2}, y_0+\tfrac{h}{2}k_1) = f(0.15, 0.725) = \boxed{1.7025}$$

$$k_3 = f(t_0+\tfrac{h}{2}, y_0+\tfrac{h}{2}k_2) = f(0.15, 0.755375) = \boxed{1.732875}$$

$$k_4 = f(t_0+h, y_0+hk_3) = f(0.3, 0.7266125) = \boxed{1.9298625}$$

$$\therefore y_1 = y_0 + \frac{0.3}{6}(k_1+2(k_2+k_3)+k_4)$$

$$\therefore y_1 = 1.015030625$$

(4) Global error

$$|e_n| \leq \frac{hk}{2M}((1+hM)^n - 1)$$

*M

$$M = \max_{[0,2]} |f_y| \quad , f(t_i, y_i) = y_i - t_i^2 + 1 \Rightarrow f_y = 1$$

$$\therefore \boxed{M = 1}$$

*K

$$k = \max_{[0,2]} |y''| \quad , \quad y = (t+1)^2 - 0.5e^t \Rightarrow y' = 2(t+1) - 0.5e^t$$

$$y'' = 2 - 0.5e^t \quad \Rightarrow \quad k = |2 - 0.5e^2| \Rightarrow \boxed{k = 1.69453}$$

$$\therefore |e_n| \leq 0.169453((1+0.2)^n - 1)$$

$e_1, e_2, \ldots, e_{10} \leftarrow$ local errors.

(2)- Use modified Euler Corrector-predictor method to approximate $y(0.3)$ up to three decimal places if $y' = x+y+1$, $y(0)=0$

Sol.

$x_0 = 0$, $y_0 = 0$ ——— $x_1 = 0.3$, $y_1 = ?$

*Predictor-step

$$y_{i+1} = y_i + h f(x_i, y_i)$$

$i=0$

$$\therefore y_1 = y_0 + 0.3 f(x_0, y_0)$$

$$\therefore y_1 = 0.3 = y_1^{c(0)}$$

*Corrector-step

$$y_{i+1}^{c(r+1)} = y_i + \frac{h}{2}\left[f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{c(r)}) \right]$$

$$\therefore y_1^{c(r+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{c(r)}) \right]$$

$$y_1^{c(r+1)} = \frac{0.3}{2}\left[1 + f(0.3, y_1^{c(r)}) \right]$$

$r=0$

$$y_1^{c(1)} = \frac{0.3}{2}\left[1 + f(0.3, y_1^{c(0)}) \right] = 0.39$$

$$e = |y_1^{c(1)} - y_1^{c(0)}| \cong \boxed{0.09}$$

$r=1$

$$y_1^{c(2)} = \frac{0.3}{2}\left[1 + f(0.3, y_1^{c(1)}) \right] = 0.4035$$

$$e = |y_1^{c(2)} - y_1^{c(1)}| \cong \boxed{0.0135}$$

$r=2$

$$y_1^{c(3)} = \frac{0.3}{2}\left[1 + f(0.3, y_1^{c(2)}) \right] = 0.405525$$

$$e = |y_1^{c(3)} - y_1^{c(2)}| = \boxed{2.025 * 10^{-3}}$$

$r=3$

$$y_1^{c(4)} = \frac{0.3}{2}\left[1 + f(0.3, y_1^{c(3)}) \right] = 0.40583$$

$$e = |y_1^{c(4)} - y_1^{c(3)}| = \boxed{3.0375 * 10^{-4}}$$

$$\therefore y(0.3) \cong 0.406$$

$i_{12} = i_{11} + 0.2[-4i_{11} + 3i_{21} + 6]$

$$i_{12} = 1.232\,A$$

$i_{22} = i_{21} + 0.2[-2.4i_{11} + 1.6\,i_{21} + 3.6]$

$$i_{22} = 1.0944\,A$$

4- Solve the 2nd order initial-Value problem $y'' - 2y' + 2y = e^t \sin t$ For $0 \le t \le 0.2$ with $y(0) = -0.4$, $y'(0) = -0.6$. Using Euler's method with $h = 0.1$

Sol.

2nd order $\Longrightarrow$ System of 1st order

let $y(t) = u_1(t)$, $y'(t) = u_2(t)$

$$\therefore \boxed{u_1' = u_2} \quad \text{— ①}$$

$$u_2' = 2y' - 2y + e^t \sin t$$

$$\therefore \boxed{u_2' = -2u_1 + 2u_2 + e^t \sin t} \quad \text{— ②}$$

*Using Euler with $h=0.1$

$t_0 = 0$	$t_1 = 0.1$	$t_2 = 0.2$
$u_{10} = -0.4$	u_{11}	u_{12}
$u_{20} = -0.6$	u_{21}	u_{22}

$u_{11} = u_{10} + 0.1[u_{20}] = -0.46$

$u_{21} = u_{20} + 0.1[-2u_{10} + 2u_{20} + e^{t_0} \sin t_0]$
$= -0.64$

$u_{12} = u_{11} + 0.1[u_{21}] = -0.524$

$u_{22} = u_{21} + 0.1[-2u_{11} + 2u_{21} + e^{t_1} \sin t_1] = -0.7402$